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Weighted differentiation Composition Operators from Fractional Cauchy Transforms to Weighted Type Spaces

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Abstract—*We characterize boundedness and compactness of weighted differentiation composition operators from fractional Cauchy transforms to weighted type spaces.*

1. INTRODUCTION

Let **D** be the open unit disk in the complex plane $\mathbb{C}$, **T** its boundary, $dA(z)$ the normalized area measure on **D** (i.e. A(**D**)=1), H^{∞} the space of all bounded holomorphic functions on **D** with the norm $||f||_{\infty} = sup_{z\in \mathbf{D}} \mid f(z) \mid$, H(**D**) the class of all holomorphic functions on **D** and $\beta_a(z)=(a-z)/(1-\bar{a}z)$, $a, z \in \mathbf{D}$ is the automorphism of **D** interchanging points a and 0. The family K_α of fractional Cauchy transforms is the collection of functions $f \in$ H(**D**) which are represented as

$$f(z)=\int_{\mathbf{T}} \frac{1}{(1-\bar{\zeta}z)^{\alpha}}\, d\mu(\zeta)\ (z \in \mathbf{D}) \qquad (1.1)$$

for some $\mu \in \mathcal{M}$, the space of all complex Borel measure on **T**. The principal branch is used in the power function in (1.1) and throughout the rest of the paper. The space K_α is a Banach space with respect to the norm

$$||f||_{K_\alpha} = inf_{\mu \in \mathcal{M}} \{||\mu||: f(z) \text{ is given by (1.1)}\} \qquad (1.2)$$

where $||\mu||$ denotes the total variation of measure μ. The space K_α may also be written as $K_\alpha = (K_\alpha)_a + (K_\alpha)_s$, where $(K_\alpha)_a$ is isometrically isomorphic to $\mathcal{M} / \overline{H}_0^1$, the closed subspace of $\mathcal{M}$ of absolutely continuous measure and $(K_\alpha)_s$ is isomorphic to $\mathcal{M}_s$ the closed subspace of $\mathcal{M}$ of singular measures.

For $f \in K_\alpha$, $|f(z)| \le ||f||_{K_\alpha} / (1-|z|)^{\alpha}$ $(z \in \mathbf{D})$ (1.3)

For more about these spaces see [1],[2], [3], [4], [6], [7], [8], [9], [10].

The weighted space $A_\nu(\mathbf{D})=A_\nu$ consists of all $f \in$ H(**D**) such that

$$||f||_{A_\nu} = sup_{z\in \mathbf{D}}\ \nu(z)|f(z)| < \infty,$$

where ν is a positive continuous function on **D** (weight). A weight ν is called typical if it is radial, i.e.

$$\nu(z) = \nu|z|, z \in \mathbf{D} \text{ decreasingly and}$$

Let φ be a holomorphic self-map of **D**. For a non-negative integer n, we define a linear operator $W^n_{\psi,\varphi}$ as follows:

$$W^n_{\psi,\varphi}\, f = \psi . f^{(n)} \circ \varphi\ , f \in \mathrm{H}(\mathbf{D})$$

It is of interest to provide a function theoretic characterization of boundedness and compactness of $W^n_{\psi,\varphi}$ from the space of Cauchy transforms to different spaces of holomorphic functions. For some recent results in this area, see [11],[12], [13] and the reference therein. In this paper, we characterize boundedness and compactness of weighted differentiation composition operators from fractional Cauchy transforms to weighted type spaces. Throughout the paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation A $\asymp$B means there is a positive constant C such that $A/C \le B \le CA$.

2. BOUNDEDNESS AND COMPACTNESS OF $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$

In this section, we characterize the boundedness and compactness of $W^n_{\psi,\varphi}$ from the space of fractional Cauchy transforms to weighted type spaces.

Theorem1. Let ν be a weight, $\alpha > 0, n \in N \cup \{0\}, \psi \in$ H(**D**) and φ a holomorphic self-map of **D**. Then $W^n_{\psi,\varphi}$: $K_\alpha \to A_\nu$ is bounded if and only if

$$M_1 := sup_{\zeta\in \mathbf{T}}\, sup_{z\in \mathbf{D}} \frac{\nu(Z)|\psi(Z)|}{|1-\bar{\zeta}\,(z)|^{n+\alpha}} < \infty. \qquad (2.1)$$

(1.3

Proof: First suppose that (2.1) holds. Let $f \in K_\alpha$. Then there is a $\mu \in \mathcal{M}$ such that $||\mu|| = ||f||_{K_\alpha}$ and

$$f(z)= \int_{\mathbf{T}} \frac{1}{(1-\bar{\zeta}z)^{\alpha}}\, d\mu(\zeta)$$

Thus, we have

$$f^n(z) = \alpha(\alpha+1)(\alpha+2)(\alpha+n-1)$$

$$\int_{\mathbf{T}} \frac{(\bar{\zeta})^n}{(1-\bar{\zeta}z)^{n+\alpha}}\, d\mu(\zeta). \qquad (2.2)$$

Replacing z in (2.2) by $\varphi(z)$, using a known inequality and multiplying such obtained inequality by $\nu(z)|\psi(z)|$, we obtain

$$\nu(z)|\psi(z)||f^n\, \varphi(z)| \leq \alpha(\alpha+1)(\alpha+2).(\alpha+n-1)$$

$$\int_{\mathbf{T}} \frac{\nu(z)|\psi(z)|}{|1-\bar{\zeta}\,\varphi(z)|^{n+\alpha}}\, d|\mu|(\zeta) \qquad (2.3)$$

$$\leq \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1)\, sup_{\zeta\in\mathbf{T}}\, sup_{z\in\mathbf{D}} \frac{\nu(z)|\psi(z)|}{|1-\bar{\zeta}\,\varphi(z)|^{n+\alpha}} \int_{\boldsymbol{T}} d|\mu|(\zeta)$$

$$= \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1)\, sup_{\zeta\in\boldsymbol{T}}\, sup_{z\in\boldsymbol{D}} \frac{\nu(z)|\psi(z)|}{|1-\bar{\zeta}\,\varphi(z)|^{n+\alpha}}\, ||\mu||$$

from which it follows that

$$\nu(z)|\psi(z)||f^n\, \varphi(z)| \leq \alpha(\alpha+1)(\alpha+2)\ldots$$

$$(\alpha+n-1) sup_{\zeta\in\boldsymbol{T}}\, sup_{z\in\boldsymbol{D}} \frac{\nu(z)|\psi(z)|}{|1-\bar{\zeta}\,\varphi(z)|^{n+\alpha}}\, ||f||_{K_\alpha}.$$

Taking the supremum over z ∈ **D,** we get

$$||W^n_{\psi,\varphi}\, f||_{A_\nu} = sup_{z\in\boldsymbol{D}}|(W^n_{\psi,\varphi}\, f)(z)| \leq \alpha$$

$$(\alpha+1)(\alpha 2)\ldots\ldots(\alpha+n-1)M_1\, ||f||_{K_\alpha}. \qquad (2.4)$$

Next suppose that $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$ is bounded. Let

$$f_\zeta(z) = \int_{\mathbf{T}} \frac{1}{(1-\bar{\zeta}z)^{\alpha}}\, d\mu(\zeta),\ \zeta \in \mathbf{T}. \qquad (2.5)$$

Then $||f_\zeta||_{k_\alpha}=1$ and

$$f^n_\zeta(z) = \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1)\ \frac{(\bar{\zeta})^n}{(1-\bar{\zeta}\,(z)^{n+\alpha}}$$

From this and the boundedness of the operator $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$, we have that $||W^n_{\psi,\varphi}\, f_\zeta\, ||_{A_\nu} \leq ||W^n_{\psi,\varphi}\, ||_{K_\alpha \to A_\nu}$, for every $\zeta \in \mathbf{T}$ and so

$$\alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1) \leq$$

$sup_{\zeta\in\boldsymbol{T}}\, sup_{z\in\boldsymbol{D}} \frac{\nu(z)|\psi(z)|}{|1-\bar{\zeta}\,\varphi(z)|^{n+\alpha}} \leq ||W^n_{\psi,\varphi}||_{K_\alpha \to A_\nu}$ and so (2.1) holds.

Theorem 2. Let ν be a weight, $\alpha > 0, n \in N \cup \{0\}$, $\psi \in$ H(**D**) , φ a holomorphic self-map of **D** and $d\lambda(z) = dA(z)$ / (1 - $|z|^2)^2$. Then $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$ is bounded if and only if

$$L_1 = sup_{\zeta\in\boldsymbol{T}}\, sup_{z\in\mathbf{D}} \int_{\mathbf{D}} \frac{|\psi(z)|^2}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}}\, \nu^2(z)\, (1-|\beta_a(z)|^2)^2 d\lambda(z) < \infty. \qquad (2.6)$$

Proof: First assume that (2.6) holds. Since ν is normal, $\nu(a) \asymp \nu(z)$ when $z \in \mathbf{D}(a,(1-|a|)\ /\ 2) = \{|z-a| < (1-|a|)\,/\,2\}$. Also it is known that $|1-\bar{a}\,z| \asymp 1-|a|^2$, for $z \in \mathbf{D}(a,(1-|a|)\ /\ 2)$**.** Using these two facts, (1.2) and the subharmonicity of the function

$$g(z) = \frac{|\psi(z)|^2}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}}$$

we obtain

$$L_1 \geq sup_{\zeta\in\boldsymbol{T}}\, sup_{a\in\mathbf{D}} \int_{D(a,(1-|a|)/2)} \frac{|\psi(z)|^2}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}}\ (1 - |\beta_a(z)|^2)^2\, d\lambda(z)$$

$$= sup_{\zeta\in\mathbf{T}}\, sup_{a\in\mathbf{D}} \int_{D(a,(1-|a|)/2)} \frac{|\psi(z)|^2}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}}\ \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\, dA(z)$$

$$\geq sup_{\zeta\in\mathbf{T}}\, sup_{a\in\mathbf{D}} \frac{\nu^2(a)|\psi(z)|^2}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}} = M_1^2. \qquad (2.7)$$

Thus by theorem 1, the operator $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$ is bounded.

Next assume that the operator $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$ is bounded. By theorem 1, we have that (2.1) holds. From this, we have

$$L_1 \leq M_1^2\, sup_{a\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\, dA(z) = M_1^2 C < \infty. \qquad (2.8)$$

The asymptotic relation $L_1 \asymp M_1^2$ follows from (2.7) and (2.8).

Proceeding as in the proof of Theorem 2, we can easily prove the following lemma.
We omit the proof.

Lemma 1. Let $\nu : \mathbf{D} \to [0,\infty)$ be a normal weight function and $d\lambda(z) = dA(z)$ / (1 - $|z|^2)^2$. Then $f \in A_\nu$ if and only if

$$I := sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f(z)|^2\, \nu^2(z)\, (1 - |\,\beta_a(z)|^2)^2\, d\lambda(z) < \infty.$$

Moreover, the following asymptotic relationship holds $||f||^2_{A_\nu} \asymp I$.

By (1.3), the unit ball B_{K_α} of K_α is a normal family, a standard argument from Proposition 3.11 in [5] yields the proof of the next lemma.

Lemma 2. Let ν be a weight, $\alpha > 0, n \in N \cup \{0\}$, $\psi \in$ H(**D**) , φ a holomorphic self-map of **D.** Then $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$ is compact if and only if any bounded sequence $\{f_m\}_{m\in\mathbb{N}}$ in K_α converging to zero on compacts subsets of **D**, we have that $\lim_{m\to\infty} ||W^n_{\psi,\varphi}\, f_m\, ||_{A_\nu}=0$.

Theorem 3. Let ν be a weight, $\alpha > 0, n \in N \cup \{0\}$, $\psi \in$ H(**D**) , φ a holomorphic self-map of **D** and $d\lambda(z) = dA(z)$ / (1 - $|z|^2)^2$ and $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$ is bounded. Then the following statements are equivalent:

1.$W^n_{\psi,\varphi} : K_\alpha \to A_\nu$ is bounded.

2.$M_3 := sup_{a\in\boldsymbol{D}} \int_{\mathbf{D}} \nu^2(z)\ (1 - |\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \infty$ and

$$\lim_{r\to 1} sup_{\zeta\in\boldsymbol{T}}\, sup_{a\in\boldsymbol{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}}\ (1 - |\beta_a(z)|^2)^2\, |\psi(z)|^2 d\lambda(z) = 0. \ (2.9)$$

Proof: (1) → (2). Since $W^n_{\psi,\varphi} : K_\alpha \to A_\nu$ is bounded, for $(z) = z^n\ /\ n! \in K_\alpha$, we get

$M_3 = sup_{a\in \mathbf{D}} \int_{\mathbf{D}} \nu^2(z)\ (1 - |\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \infty.$

Let $f_m(z)=z^m, m \in \mathbb{N}$. It is norm bounded sequence in K_α converging to zero uniformly on compact subsets of **D**. Hence by Lemma 2, it follows that $||W^n_{\psi,\varphi} f_\zeta||_{A_\nu} \to 0$ as $m \to \infty$. Thus for every $\epsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, we have

$$(\prod_{j=0}^n (m-j))^2\ sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\varphi(z)|^{2(m-n)}\ \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \epsilon. \quad (2.10)$$

From (2.9), we have that for each $r \in (0,1)$

$$r^{2(m-n)}\ (\prod_{j=0}^n (m-j))^2\ sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \epsilon \quad (2.11)$$

Hence for $\in [\prod_{j=0}^n (m-j)^{\frac{-1}{m-n}}, 1]$ we have

$$sup_{a\in D} \int_{|\varphi(z)|>r} \nu^2(z)(1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \epsilon \quad (2.12)$$

Let $f \in B_{K_\alpha}$ and $f_t(z), 0 < t < 1$.

Then $sup_{0<t<1} ||f_t||_{K_\alpha} \leq ||f||_{K_\alpha}$, $f_t \in K_\alpha$, $t \in (0,1)$ and $f_t \to f$ uniformly on compacts subset of **D** as $t \to 1$. The compactness of $W^n_{\psi,\varphi}: K_\alpha \to A_\nu$ implies that $\lim_{t\to 1} ||W^n_{\psi,\varphi} f_t - W^n_{\psi,\varphi} f||_{A_\nu}=0$. Hence for every $\epsilon > 0$, there is a $t \in (0,1)$ such that

$$sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} |f_t^n(\varphi(z)) - f^n(\varphi(z))|^2\ \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \epsilon \quad (2.13)$$

By inequalities (2.12) and (2.13), we have

$$sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} |f^n(\varphi(z))|^2\ \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z)$$
$$\leq 2\, sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f_t^n(\varphi(z)) - f^n(\varphi(z))|^2\ \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) + 2\, sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} |f^n(\varphi(z))|^2\ \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) \leq 2\epsilon(1+||f_t^n||^2_\infty).$$

Hence for every $f \in B_{K_\alpha}$, there is a $\delta_0 \in (0,1)$, $\delta_0 = \delta_0(f,\epsilon)$, such that for $r \in (\delta_0, 1)$

$$sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} |f^n(\varphi(z))|^2\ \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \epsilon.$$

From the compactness of $W^n_{\psi,\varphi}: K_\alpha \to A_\nu$, we have that for every $\epsilon > 0$ there is a finite collection of functions $f_1, f_2, f_3, \ldots\ldots, f_k \in B_{K_\alpha}$ such that for each $f \in B_{K_\alpha}$, there is a $j \in \{1,2,3,\ldots.k\}$ such that

$$sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f^n(\varphi(z)) - f_j^n(\varphi(z))|^2\ \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \epsilon \quad (2.14)$$

On the other hand, from (2.14) it follows that if $\delta := max_{1\leq j\leq k}\ \delta_j(f_j,\epsilon)$, then for $r \in (\delta, 1)$ and all $j \in \{1,2,3,\ldots.k\}$ we have

$$sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} f_j^n(\varphi(z)|)^2\ \nu^2(z)\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < \epsilon \quad (2.15)$$

From (2.14) and (2.15), we have that for $r \in (\delta, 1)$ and every $f \in B_{K_\alpha}$

$$sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} |f^n(\varphi(z))^2\ \nu^2(z)\ (1-|\beta_a(z))^2|^2\ |\psi(z)|^2 d\lambda(z) < 4\epsilon. \quad (2.16)$$

Applying (2.16) to the functions $f_\zeta(z) = 1/(1-\bar{\zeta}z)^\alpha$, $\zeta \in \mathbf{T}$, we obtain

$$sup_{\zeta\in\mathbf{T}}\, sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\bar{\zeta}\varphi(z)|^{2(n+\alpha)}}\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2 d\lambda(z) < 4\epsilon / (\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1))^2$$

from which (2.9) follows.

(2) $\Rightarrow$ (1). Assume that $\{f_m\}_{m\in\mathbb{N}}$ is a bounded sequence in K_α, say by L, converging to 0 uniformly on compacts of **D** as $m \to \infty$. Then by the Weierstrass's theorem, $f_m^{(k)}$ also converges to 0 uniformly on compacts of **D**, for each $k \in \mathbb{N}$. We need to show that $||W^n_{\psi,\varphi} f_m||_{A_\nu} \to 0$ as $m \to \infty$. For each $m \in \mathbb{N}$, we can find a $\mu_m \in$ ꞵ with $||\mu_m||=||\mu_m||_{K_\alpha}$ such that

$$f_m(z) = \int_{\mathbf{T}} \frac{1}{(1-\bar{\zeta}z)^\alpha}\, d\mu_m(\zeta) \quad (2.17)$$

Differentiating (2.17) n times, composing such obtained equation by φ, applying Jensen's inequality, as well as the boundedness of sequence $\{f_m\}_{m\in\mathbb{N}}$, we obtain

$$|f_m^{(n)}(\varphi(w))|^2 \leq L(\alpha(\alpha+1)(\alpha+2)\ldots.(\alpha+n-1))2\int_{\mathbf{T}} 1/|1-\zeta\varphi w|\,2(n+\alpha)\, d|\mu m|(\zeta) \quad (2.18)$$

By the second condition in (2), we have that for every $\epsilon > 0$ there is an $r_1 \in (0,1)$ such that for $r \in (r_1, 1)$, we have

$$sup_{\zeta\in\mathbf{T}}\, sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\bar{\zeta}\varphi(z)|^{2(n+\alpha)}}\ (1-|\beta_a(z)|^2)^2 |\psi(z)|^2 d\lambda(z) < \epsilon \quad (2.19)$$

By (1.3), we have

$$||W^n_{\psi,\varphi} f_m||_{A_\nu} \asymp sup_{a\in\mathbf{D}} \int_{|\varphi(z)|\leq r} |f_m^{(n)}(\varphi(w)|)^2\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2\ \nu^2(z) d\lambda(z)$$
$$+\ sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} |f_m^{(n)}(\varphi(w)|)^2\ (1-|\beta_a(z)|^2)^2\ |\psi(z)|^2\ \nu^2(z) d\lambda(z).$$

Using first condition in (2), (2.19),Fubini's theorem and the fact that $sup_{|w|\leq r} |f_m^{(n)}(w)|^2 < \epsilon$,

for sufficiently large m, say $m \geq m_0$, we have that

$$||W_{\psi,\varphi}^{n} f_m \,||_{A_\nu} \leq sup_{\varphi(z)\leq r} \;|\; f_m^{(n)}(\varphi(w))|^2 \; sup_{a\in D} \int_{|\varphi(z)|\leq r}(1-|\beta_a\,(z)|^2)^2\, |\psi(z)|^2\, \nu^{\,2}(z)d\lambda(z)$$

$$+ \; sup_{a\in\mathbf{D}} \int_{\mathbf{T}}\int_{|\varphi(z)|>r} \frac{\nu^{\,2}\,(z)}{|1-\bar{\zeta}\,\varphi(z)|^{\,2(n+\alpha)}}\,(1-|\beta_a\,(z)|^2)^2\, |\psi(z)|^2 d\lambda(z)\; d|\mu_{\mathrm{m}}|\,(\zeta)$$

$$\leq \;(M_3 \;+\; \int_{\mathbf{T}} d|\mu_{\mathrm{m}}|\,(\zeta)) \leq (M_3 + L)\,\epsilon.$$

Since ϵ is an arbitrary, the result follows by Lemma 2.

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